

# Posimodular Function Optimization

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## Abstract

Given a posimodular function  $f : 2^V \rightarrow \mathbb{R}$  on a finite set  $V$ , we consider the problem of finding a nonempty subset  $X$  of  $V$  that minimizes  $f(X)$ . Posimodular functions often arise in combinatorial optimization such as undirected cut functions. In this paper, we show that any algorithm for the problem requires  $\Omega(2^{\frac{n}{7.54}})$  oracle calls to  $f$ , where  $n = |V|$ . It contrasts to the fact that the submodular function minimization, which is another generalization of cut functions, is polynomially solvable.

When the range of a given posimodular function is restricted to be  $D = \{0, 1, \dots, d\}$  for some nonnegative integer  $d$ , we show that  $\Omega(2^{\frac{d}{15.08}})$  oracle calls are necessary, while we propose an  $O(n^d T_f + n^{2d+1})$ -time algorithm for the problem. Here,  $T_f$  denotes the time needed to evaluate the function value  $f(X)$  for a given  $X \subseteq V$ .

We also consider the problem of maximizing a given posimodular function. We show that  $\Omega(2^{n-1})$  oracle calls are necessary for solving the problem, and that the problem has time complexity  $\Theta(n^{d-1} T_f)$  when  $D = \{0, 1, \dots, d\}$  is the range of  $f$  for some constant  $d$ .

**Keyword:** Posimodular function, Algorithm, Horn CNF, Extreme sets

## 1 Introduction

Let  $V$  denote a finite set with  $n = |V|$ . A set function  $f : 2^V \rightarrow \mathbb{R}$  is called *posimodular* if

$$f(X) + f(Y) \geq f(X \setminus Y) + f(Y \setminus X) \quad (1.1)$$

for all  $X, Y \subseteq V$ , where  $\mathbb{R}$  denotes the set of all reals. Posimodularity is one of the most fundamental and important properties in combinatorial optimization [5, 7, 11, 13, 14, 17]. Typically, it is a key for efficient solvability of undirected network optimization and the related problems, since cut functions for undirected networks are posimodular. Note that cut functions for directed networks are not posimodular. We can observe that posimodularity helps to create complexity gaps for a number of network optimization problems, in the sense that the undirected versions can be solved faster than the directed versions. For example, the local edge-connectivity augmentation problem in undirected networks is polynomially solvable, but the problem in directed networks is NP-hard [4]. As for the source location problem with uniform demands or with uniform costs, the undirected versions can be solved in polynomial time [1, 18], while the directed versions are NP-hard [8]. More generally, the currently fastest algorithm for minimizing a submodular and posimodular function achieves

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$O(n^3 T_f)$  time [12], while the one for minimizing a submodular function achieves  $O(n^5 T_f + n^6)$  time [15], where a set function  $f : 2^V \rightarrow \mathbb{R}$  is called *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \quad (1.2)$$

for all  $X, Y \subseteq V$ , and  $T_f$  denotes the time needed to evaluate the function value  $f(X)$  for a given  $X \subseteq V$ . One of the reasons for these phenomena is based on the following two structural properties on posimodular functions.

A subset  $X$  of  $V$  is called *extreme* if every nonempty proper subset  $Y$  of  $X$  satisfies  $f(Y) > f(X)$ . It is known that the family  $\mathcal{X}(f)$  of extreme sets is *laminar* (i.e., every two members  $X$  and  $Y$  in  $\mathcal{X}(f)$  satisfy  $X \cap Y = \emptyset$ ,  $X \subseteq Y$ , or  $X \supseteq Y$ ), when  $f$  is posimodular. Note that if  $X, Y \in \mathcal{X}(f)$  would satisfy  $X \cap Y, X \setminus Y, Y \setminus X \neq \emptyset$ , then we have  $f(X) + f(Y) \geq f(X \setminus Y) + f(Y \setminus X) > f(X) + f(Y)$ , a contradiction. The family  $\mathcal{X}(f)$  of extreme sets for an undirected cut function  $f$  represents the connectivity structure of a given network and helps to design many efficient network algorithms [9, 20]. For example, the undirected source location problem with uniform demands can be solved in  $O(n)$  time, if the family  $\mathcal{X}(f)$  is known in advance, where  $n$  corresponds to the number of vertices in the network [10]. In fact,  $\mathcal{X}(f)$  can be computed in  $O(n(m + n \log n))$  time for any undirected cut function [10], where  $m$  denotes the number of edges in the network. We note that  $\mathcal{X}(f)$  can be found in  $O(n^3 T_f)$  time if  $f$  is posimodular and submodular [11].

The other structural property is for solid sets. For an element  $v \in V$ , a subset  $X$  of  $V$  is called *v-solid set* if  $v \in X$  and every nonempty proper subset  $Y$  of  $X$  that contains  $v$  satisfies  $f(Y) > f(X)$ . Let  $\mathcal{S}(f)$  denote the family of all solid sets, i.e.,  $\mathcal{S}(f) = \bigcup_{v \in V} \{v\text{-solid } X\}$ . It is known [17] that the family  $\mathcal{S}(f)$  forms a tree hypergraph if  $f$  is posimodular. Similarly to the previous case for  $\mathcal{X}(f)$ , if a host tree  $T$  of  $\mathcal{S}(f)$  is known in advance, this structure enables us to construct a polynomial time algorithm for the minimum transversal problem for posimodular functions  $f$ , which is an extension of the undirected source location problem with uniform costs [18] and the undirected external network problem [19]. If  $f$  is in addition submodular, a host tree  $T$  can be computed in polynomial time.

We here remark that these structural properties on  $\mathcal{X}(f)$  and  $\mathcal{S}(f)$  follow from the posimodularity of  $f$ , and that the submodularity is needed to derive such structures efficiently, more precisely, the submodularity is assumed due to the property that  $\min\{f(X) \mid \emptyset \neq X \subseteq V\}$  can be computed in polynomial time.

On the other hand, to our best knowledge, all the previous results for the posimodular optimization also make use of the submodularity or symmetricity, since undirected cut functions, the most representative posimodular functions, are also submodular and symmetric. Here a set function  $f : 2^V \rightarrow \mathbb{R}$  is called *symmetric* if  $f(X) = f(V \setminus X)$  holds for any  $X \subseteq V$ . We note that a function is symmetric posimodular if and only if it is symmetric submodular, since the symmetricity of  $f$  implies that  $f(X) + f(Y) = f(V \setminus X) + f(Y)$  and  $f(X \setminus Y) + f(Y \setminus X) = f((V \setminus X) \cup Y) + f((V \setminus X) \cap Y)$ .

In this paper, we focus on the posimodular function minimization defined as follows.

#### POSIMODULAR FUNCTION MINIMIZATION

Input: A posimodular function  $f : 2^V \rightarrow \mathbb{R}$ , (1.3)

Output: A nonempty subset  $X^*$  of  $V$  such that  $f(X^*) = \min_{X \subseteq V: X \neq \emptyset} f(X)$ .

Here an input  $f$  is given by an oracle that answers  $f(X)$  for a given subset  $X$  of  $V$ , and we assume that the optimal value  $f(X^*)$  is also output. The problem was posed as an open problem on the Egres open problem list [3] in 2010, as the negamodular function maximization,

where a set function  $f$  is *negamodular*, if  $-f$  is posimodular. We also consider the posimodular function maximization, as the submodular function maximization has been intensively studied in recent years.

## Our Contributions

The main results obtained in this paper can be summarized as follows.

1. We show that any algorithm for the posimodular function minimization requires  $\Omega(2^{\frac{n}{7.54}})$  oracle calls.
2. For a nonnegative integer  $d$ , let  $D = \{0, 1, \dots, d\}$  denote the range of  $f$ , i.e.,  $f : 2^V \rightarrow D$ . Then we show that  $\Omega(2^{\frac{d}{15.08}})$  oracle calls are necessary for the posimodular function minimization, while we propose an  $O(n^d T_f + n^{2d+1})$ -time algorithm for the problem. Also, as its byproduct, the family  $\mathcal{X}(f)$  of all extreme sets can be computed in  $O(n^d T_f + n^{2d+1})$  time. Furthermore, we show that all optimal solutions can be generated with  $O(n T_f)$  delay after generating all locally minimal optimal solutions in  $O(n^d T_f + n^{2d+1})$  time.
3. We show that the posimodular function maximization requires  $\Omega(2^{n-1})$  oracle calls, and that the problem has time complexity  $\Theta(n^{d-1} T_f)$  when  $D = \{0, 1, \dots, d\}$  is the range of  $f$  for some constant  $d$ .

The first result contrasts to the submodular function minimization, which can be solved in polynomial time, and the second result implies the polynomiality for the posimodular function minimization if the range is bounded. The last result shows that the posimodular function maximization is also intractable.

The rest of this paper is organized as follows. Section 2 presents basic definitions and preparatory properties on posimodular functions. In Section 3, we show the hardness results for the posimodular function minimization. In Section 4, we propose an  $O(n^d T_f + n^{2d+1})$ -time algorithm for the posimodular function minimization when  $D$  is the range of  $f$ . We also consider the problems for computing all extreme sets and all optimal solutions. Section 5 treats the posimodular function maximization.

## 2 Preliminaries

Let  $V$  be a finite set with  $n = |V|$ . For two subsets  $X, Y$  of  $V$ , we say that  $X$  and  $Y$  *intersect each other* if each of  $X \setminus Y$ ,  $Y \setminus X$ , and  $X \cap Y$  is nonempty. Let  $f : 2^V \rightarrow \mathbb{R}$  be a posimodular function. Notice that any posimodular function  $f$  satisfies

$$f(X) \geq f(\emptyset) \text{ for all } X \subseteq V, \quad (2.1)$$

since  $f(X) + f(X) \geq f(\emptyset) + f(\emptyset)$ . Throughout the paper, we assume that  $f(\emptyset) = 0$ , since otherwise, we can replace  $f(X)$  by  $f(X) - f(\emptyset)$  for all  $X \subseteq V$ .

We here show a preparatory lemma for posimodular functions.

**Lemma 2.1** *For a posimodular function  $f : 2^V \rightarrow \mathbb{R}$ , let  $T$  be a subset of  $V$  with  $f(T) = \max\{f(X) \mid X \subseteq V\}$ . For a nonempty proper subset  $U$  of  $V$ , the following two properties hold.*

- (i) *If  $U \cap T = \emptyset$ , then we have  $f(U) \geq f(\{v\})$  for any  $v \in U$ .*

(ii) If  $U \supseteq T$ , then we have  $f(U) \geq f(\{v\})$  for any  $v \notin U$ .

**Proof.** If  $T = V$ , then the statements (i) and (ii) of the lemma clearly hold, since no nonempty proper subset  $U$  of  $V$  satisfies  $U \cap T = \emptyset$  or  $U \supseteq T$ . On the other hand, if  $T = \emptyset$ , then we have  $f(X) = 0$  for all  $X$  by (2.1) and the assumption on  $f$ . This again implies the statements of the lemma. We therefore assume that  $T$  is a nonempty proper subset of  $V$ .

For a nonempty subset  $U$  with  $U \cap T = \emptyset$ , let  $v$  be an element in  $U$ . Then by (1.1), we have  $f(U) + f(T \cup (U \setminus \{v\})) \geq f(T) + f(\{v\})$ . Since  $T$  is a maximizer of  $f$ ,  $f(U) \geq f(\{v\})$  holds, which proves (i) of the lemma. For a proper subset  $U$  with  $U \supseteq T$ , let  $v$  be an element in  $V \setminus U$ . Then by (1.1), we have  $f(U) + f((U \setminus T) \cup \{v\}) \geq f(T) + f(\{v\})$ . Since  $T$  is a maximizer of  $f$ ,  $f(U) \geq f(\{v\})$  holds, which proves (ii) of the lemma.  $\square$

In this paper, we sometimes utilize a Boolean function  $\varphi : \{0, 1\}^V \rightarrow \{0, 1\}$ . Let  $x_v$  ( $v \in V$ ) be a Boolean variable, and a *literal* means a Boolean variable  $x_v$  or its complement  $\bar{x}_v$ . A disjunction of literals  $c = \bigvee_{v \in P(c)} x_v \vee \bigvee_{i \in N(c)} \bar{x}_i$  is called a *clause* if  $P(c) \cap N(c) = \emptyset$ , and a *conjunctive normal form* (CNF, in short) is a conjunction of clauses. A CNF is called *Horn*, *definite Horn*, and *dual Horn* if each clause has at most one positive literal, exactly one positive literal, and at most one negative literal, respectively.

### 3 Hardness of the posimodular function minimization

In this section, we analyze the number of oracle calls necessary to solve the posimodular function minimization.

Let  $g : 2^V \rightarrow \mathbb{R}_+$  be a function defined by  $g(X) = |X|$  if  $X \neq \emptyset$ , and  $g(\emptyset) = 0$ . Clearly,  $g$  is posimodular, since  $g$  is monotone, i.e.,  $g(X) \geq g(Y)$  holds for all two subsets  $X$  and  $Y$  of  $V$  with  $X \supseteq Y$ . For a positive integer  $k$  with  $k \leq n/2$ , let  $S$  be a subset of  $V$  of size  $|S| = 2k$ . Define a function  $g_S : 2^V \rightarrow \mathbb{R}_+$  by

$$g_S(X) = \begin{cases} 2k - |X| & \text{if } X \subseteq S \text{ and } |X| \geq k + 1, \\ g(X) & \text{otherwise.} \end{cases}$$

We can see that  $g_S$  is a posimodular function close to  $g$ .

**Claim 3.1**  $g_S$  is posimodular.

**Proof.** Note first that  $g_S(X) \leq |X|$  for all  $X \subseteq V$ , since  $|X| - (2k - |X|) \geq 0$  if  $|X| \geq k + 1$ . Let  $X$  and  $Y$  be two subsets of  $V$  with  $X \cap Y \neq \emptyset$ . We separately consider the following two cases.

If at least one of  $X$  and  $Y$  has the identical function values for  $g_S$  and  $g$ , say  $g_S(X) = g(X)$ , then we have  $g_S(X) - g_S(X \setminus Y) \geq |X \cap Y|$ . If  $g_S(Y) = g(Y)$  is also satisfied, then we obtain  $g_S(Y) - g_S(Y \setminus X) \geq |X \cap Y|$ , and hence the posimodular inequality (1.1) holds for such  $X$  and  $Y$ . On the other hand, if  $g_S(Y) \neq g(Y)$ , i.e.,  $Y \subseteq S$  and  $|Y| \geq k + 1$ , then we have  $g_S(Y) - g_S(Y \setminus X) \geq -|X \cap Y|$ , which again implies the posimodular inequality (1.1).

If  $g_S(X) \neq g(X)$  and  $g_S(Y) \neq g(Y)$  are satisfied, then we have  $g_S(X) = 2k - |X|$  and  $g_S(Y) = 2k - |Y|$ . Since  $|X \setminus Y|, |Y \setminus X| \leq k$ , we also have  $g_S(X \setminus Y) = |X \setminus Y|$  and  $g_S(Y \setminus X) = |Y \setminus X|$ . Hence, it holds that

$$g_S(X) + g_S(Y) - (g_S(X \setminus Y) + g_S(Y \setminus X)) = 4k - 2|X \cup Y| \geq 0,$$

where the last inequality follows from  $X \cup Y \subseteq S$  and  $|S| = 2k$ . Therefore the posimodular inequality (1.1) holds.  $\square$

Let  $\mathcal{G} = \{g\} \cup \{g_S \mid S \subseteq V, |S| = 2k\}$ . We below show that exponential oracles is necessary to distinguish among posimodular functions in  $\mathcal{G}$ .

Let  $\mathcal{S} = \{S \subseteq V \mid |S| = 2k\}$  and  $\mathcal{T} = \{T \subseteq V \mid k+1 \leq |T| \leq 2k\}$ . Consider the following integer programming problem:

$$\begin{aligned} & \text{minimize} && \sum_{T \in \mathcal{T}} z_T \\ & \text{subject to} && \sum_{T \in \mathcal{T}: T \subseteq S} z_T \geq 1 \quad \text{for each } S \in \mathcal{S}, \\ & && z_T \in \{0, 1\} \quad \text{for each } T \in \mathcal{T}. \end{aligned} \tag{3.1}$$

Note that any posimodular function  $f$  in  $\mathcal{G}$  satisfies  $f(X) = g(X)$  if  $|X| \leq k$  or  $|X| \geq 2k+1$ . Oracle calls for such sets  $X$  do not help to distinguish among posimodular functions in  $\mathcal{G}$ . Therefore, we can restrict our attention to subsets  $T$  in  $\mathcal{T}$  for oracle calls.

**Lemma 3.2** *Let  $q_k$  denote the optimal value for (3.1). Then at least  $q_k$  oracle calls is necessary to distinguish among posimodular functions in  $\mathcal{G}$ .*

**Proof.** Assume to the contrary that there exists an algorithm  $A$  which distinguishes by at most  $q_k - 1$  oracle calls among posimodular functions in  $\mathcal{G}$ . Let  $\mathcal{X}$  denote the family of subsets of  $V$  which are called by  $A$  if a posimodular function  $g$  is an input of  $A$ . Since  $|\mathcal{X}| \leq q_k - 1$ , we have a subset  $S$  in  $\mathcal{S}$  such that no  $X \in \mathcal{X}$  satisfies  $X \subseteq S$  and  $|X| \geq k+1$ . This means that  $g_S(X) = g(X)$  for all  $X \in \mathcal{X}$ , which contradicts that  $A$  distinguishes between  $g$  and  $g_S$ .  $\square$

It follows from Lemma 3.2 that  $q_k$  oracle calls are required for the posimodular function minimization. We now analyze the optimal value  $q_k$  for (3.1).

**Lemma 3.3** *Let  $q_k$  denote the optimal value for (3.1). Then we have  $q_k \geq \binom{n}{k+1} / \binom{2k}{k+1}$ .*

**Proof.** Consider the linear programming relaxation for Problem (3.1) which is obtained by replacing each binary constraint  $z_T \in \{0, 1\}$  by  $z_T \geq 0$ :

$$\begin{aligned} & \text{minimize} && \sum_{T \in \mathcal{T}} z_T \\ & \text{subject to} && \sum_{T \in \mathcal{T}: T \subseteq S} z_T \geq 1 \quad \text{for each } S \in \mathcal{S}, \\ & && z_T \geq 0 \quad \text{for each } T \in \mathcal{T}. \end{aligned} \tag{3.2}$$

Define a vector  $z^* \in \mathbb{R}^{\mathcal{T}}$  by  $z_T^* = 1/\binom{2k}{k+1}$  if  $|T| = k+1$ , and 0 otherwise. Note that  $z^*$  is feasible to (3.2), and the objective value is

$$\sum_{T \in \mathcal{T}} z_T^* = \frac{\binom{n}{k+1}}{\binom{2k}{k+1}}. \tag{3.3}$$

Moreover, we show that it is optimal to (3.2).

Define  $y \in \mathbb{R}^{\mathcal{S}}$  by  $y_S = 1/\binom{n-(k+1)}{k-1}$  for all  $S \in \mathcal{S}$ . Then this  $y$  is feasible to the dual problem of (3.2), and the objective value is

$$\sum_{S \in \mathcal{S}} y_S = \frac{\binom{n}{2k}}{\binom{n-(k+1)}{k-1}} = \frac{\binom{n}{k+1}}{\binom{2k}{k+1}}. \tag{3.4}$$

By (3.3) and (3.4),  $z^*$  is an optimal solution of (3.2). Since it is a relaxation of the minimization problem, we have  $q_k \geq \binom{n}{k+1} / \binom{2k}{k+1}$ .  $\square$

For  $k \geq 2$ , we note that

$$\begin{aligned}
\frac{\binom{n}{k+1}}{\binom{2k}{k+1}} &= \frac{n!(k-1)!}{(2k)!(n-k-1)!} \\
&\geq \frac{2\pi}{e^2} \cdot \frac{n^{n+1/2}(k-1)^{k-1/2}}{(2k)^{2k+1/2}(n-k-1)^{n-k-1/2}} \\
&\geq \frac{2\pi}{e^2} \cdot \left(\frac{n}{2k}\right)^{k+1} \cdot \left(\frac{k-1}{2k}\right)^{k-1/2} \\
&\geq \frac{2\pi}{e^2} \cdot \left(\frac{n}{4k}\right)^{k+1}.
\end{aligned} \tag{3.5}$$

Here the second, third, and fourth inequalities respectively follow from Stirling's inequalities  $\sqrt{2\pi}n^{n+1/2}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}$ ,  $n \geq n-k-1$ , and  $(1 - \frac{1}{k})^{k-1/2} \geq \frac{1}{2\sqrt{2}}$  for  $k \geq 2$ . By setting  $n = \lceil 4ek \rceil$ , we obtain that (3.5) is  $\Omega(e^{\frac{n}{4e}}) = \Omega(2^{\frac{n}{7.54}})$ .

Thus, we have the following theorem.

**Theorem 3.4** *Any algorithm for the posimodular function minimization requires  $\Omega(2^{\frac{n}{7.54}})$  oracle calls.*

Let us next consider the case in which the range of  $f$  is bounded by  $D = \{0, 1, \dots, d\}$  for some nonnegative integer  $d$ . We show the exponential lower bound in a similar way to the proof of Theorem 3.4.

Let  $T$  be a subset of  $V$  with  $|T| = \lfloor d/2 \rfloor$ . Define  $g : 2^V \rightarrow D$  by

$$g(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ |X| & \text{if } \emptyset \neq X \subseteq T, \\ |T| + |T \cap X| & \text{otherwise.} \end{cases}$$

For a positive integer  $k$  with  $2k \leq |T|$ , let  $S$  be a subset of  $T$  with  $|S| = 2k$ . Define a function  $g_S : 2^V \rightarrow D$  by

$$g_S(X) = \begin{cases} 2k - |X| & \text{if } X \subseteq S \text{ and } |X| \geq k+1, \\ g(X) & \text{otherwise.} \end{cases}$$

The monotonicity of  $g$  implies that  $g$  is posimodular. The posimodularity of  $g_S$  can be shown as follows.

Let  $X$  and  $Y$  be two subsets of  $V$ . If both  $X$  and  $Y$  are subsets of  $T$ , then the posimodular inequality (1.1) follows from Claim 3.1. We therefore assume that  $X \setminus T \neq \emptyset$ . Then  $g_S(X) = |T| + |T \cap X|$  holds. Note that  $|T| + |T \cap Z| \geq g_S(Z)$  holds for all  $Z \subseteq T$ . Thus we have

$$\begin{aligned}
g_S(X) - g_S(X \setminus Y) &\geq |T \cap X| - |T \cap (X \setminus Y)| \\
&= |T \cap X \cap Y| \geq 0.
\end{aligned} \tag{3.6}$$

If  $Y \setminus T \neq \emptyset$  is also satisfied, then we have  $g_S(Y) \geq g_S(Y \setminus X)$ , from which the posimodular inequality (1.1) holds. On the other hand, if  $Y \subseteq T$ , then we have  $g_S(Y) - g_S(Y \setminus X) \geq -|X \cap Y|$  by  $Y \setminus X \subseteq T$ . Moreover, (3.6) implies  $g_S(X) - g_S(X \setminus Y) \geq |X \cap Y|$  by  $X \cap Y \subseteq T$ , and hence we obtain (1.1).

If  $k \geq 2$  and  $|T| \approx 4ek$  (and hence  $d \approx 8ek$ ), then by applying an argument similar to Lemmas 3.2 and 3.3, we have the following result.

**Theorem 3.5** *Assume that a given posimodular function has range  $D$ . Then the posimodular function minimization requires  $\Omega(2^{\frac{d}{15.08}})$  oracle calls.*

## 4 Polynomial time algorithm for posimodular function minimization when $d$ is a constant

In this section, we show that the posimodular function minimization can be solved in polynomial time if an input posimodular function is restricted to be  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  for some constant  $d$ . We first show that for  $d \leq 3$ , the posimodular function minimization can be solved efficiently by repeatedly contracting semi-extreme sets, and then provides an  $O(n^d T_f + n^{2d+1})$ -time algorithm for general  $d$ .

In this section, an optimal solution to the posimodular function minimization (1.3) is referred to as a *minimizer of  $f$*  (among nonempty subsets).

### 4.1 Case in which $d \in \{0, 1, 2, 3\}$

Let  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  be a function, and let  $s$  be an element with  $s \notin V$ . For a subset  $S \subseteq V$ , let  $f' : 2^{(V \setminus S) \cup \{s\}} \rightarrow \{0, 1, \dots, d\}$  be a function defined by

$$f'(X) = \begin{cases} f(X) & \text{if } s \notin X, \\ f((X \setminus \{s\}) \cup S) & \text{otherwise.} \end{cases}$$

We say that *the function  $f'$  is obtained from  $f$  by contracting a subset  $S$  of  $V$  into an element  $s$* . Notice that  $f'$  is posimodular if it is obtained from a posimodular function by contraction. A nonempty subset  $X$  of  $V$  is called *semi-extreme* (w.r.t  $f$ ) if all nonempty subsets  $Y$  of  $X$  satisfy  $f(Y) \geq f(X)$ . By the following lemma, we can contract any semi-extreme set while keeping at least one minimizer of  $f$ .

**Lemma 4.1** *Let  $f$  be a posimodular function. For any semi-extreme set  $X$ , there exists a minimizer  $Y$  of  $f$  such that  $Y \supseteq X$  or  $X \cap Y = \emptyset$ .*

**Proof.** Assume that a minimizer  $Y$  of  $f$  satisfies  $Y \not\supseteq X$  and  $X \cap Y \neq \emptyset$ . If  $Y$  is a subset of  $X$ , then  $X$  is also a minimizer of  $f$  by the semi-extremeness of  $X$ . On the other hand, if  $Y$  intersects  $X$ , then it follows from (1.1) that  $f(X) + f(Y) \geq f(X \setminus Y) + f(Y \setminus X)$ . Since  $X$  is semi-extreme, we have  $f(X) \leq f(X \setminus Y)$ . It follows that  $f(Y) \geq f(Y \setminus X)$ , which implies that  $Y \setminus X$  is also a minimizer of  $f$ .  $\square$

The following lemma indicates that we can obtain a minimizer of  $f$  after contracting a subset  $X$  of  $V$  with  $|X| = 2$  at most  $n$  times.

**Lemma 4.2** *If  $d \leq 3$ , then there exists a semi-extreme set  $X$  with  $|X| = 2$ , or a minimizer  $Y$  of  $f$  with  $|Y| = 1$  or  $|Y| \geq n - 1$ .*

**Proof.** Consider the case in which  $n \geq 4$ , since the lemma clearly holds for  $n \leq 3$ . Assume to the contrary that no subset  $X$  with  $|X| = 2$  is semi-extreme and no subset  $Y$  with  $|Y| = 1$ ,  $n - 1$  or  $n$  is a minimizer of  $f$ . Let  $X^*$  be a minimizer of  $f$ . Then by the assumption, we have

$$\begin{aligned} f(Y) &\geq f(X^*) + 1 (\geq 1) && \text{for all subsets } Y \text{ with } |Y| = 1, n - 1 \text{ or } n, \\ f(X) &\geq f(X^*) + 2 (\geq 2) && \text{for all subsets } X \text{ with } |X| = 2. \end{aligned} \tag{4.1}$$

This already proves this lemma for  $d = 1$ .

If  $d = 2$ , then it follows from (4.1) that all subsets  $X$  with  $|X| = 2$  satisfy  $f(X) = 2$ . Hence, by Lemma 2.1, any nonempty proper subset  $Z$  of  $V$  satisfies  $f(Z) \geq \min\{f(v) \mid v \in$

$V\}$ . This implies that some element of  $V$  or  $V$  is a minimizer of  $f$ , which contradicts the assumption.

For  $d = 3$ , we separately consider the cases in which the optimal value  $f(X^*)$  is 0, 1, and at least 2.

**Case**  $f(X^*) \geq 2$ . By (4.1) we have  $f(X) \geq 4$  for all subsets  $X$  of  $V$  with  $|X| = 2$ , which contradicts the fact that  $d = 3$ .

**Case**  $f(X^*) = 0$ . By the assumption, we have  $|X^*| \geq 3$ . Moreover, if  $|X^*| \leq n - 2$ , then there exists a subset  $Z$  of  $V$  such that  $|X^* \setminus Z| = |Z \setminus X^*| = 2$ . By applying (1.1) to  $X^*$  and  $Z$ , we have  $3 \geq f(X^*) + f(Z) \geq f(X^* \setminus Z) + f(Z \setminus X^*)$ , from which  $f(X^* \setminus Z) \leq 1$  or  $f(Z \setminus X^*) \leq 1$ . Since this contradicts (4.1), we have  $|X^*| \geq n - 1$ , which again contradicts (4.1).

**Case**  $f(X^*) = 1$ . By (4.1), all subsets  $X$  of  $V$  with  $|X| = 2$  satisfy  $f(X) = 3$ . Similarly to the case of  $d = 2$ , Lemma 2.1 implies that any nonempty proper subset  $Z$  of  $V$  satisfies  $f(Z) \geq \min\{f(v) \mid v \in V\}$ . Hence some element of  $V$  or  $V$  is a minimizer of  $f$ , which contradicts the assumption.  $\square$

By the lemma, for  $d \leq 3$ , we first check function values  $f(X)$  for all subsets  $X$  with  $|X| = 1, 2, n - 1$ , and  $n$ . If no subset  $X$  with  $|X| = 2$  is semi-extreme, then we output a subset  $X^*$  which satisfies  $f(X^*) = \min_{X: |X|=1, n-1, \text{ or } n} f(X)$ . Otherwise (i.e., if some  $X$  with  $|X| = 2$  is semi-extreme), we consider the function  $f'$  obtained from  $f$  by contracting  $X$  into a new element  $x$ , and check  $f'(X')$  for all subsets  $X'$  with  $|X'| = 1, 2, n - 2$ , and  $n - 1$ . Note that it is enough to check  $f(X')$  for subsets  $X'$  with  $X' \ni x$  and  $|X'| = 2$ , since the other  $X'$  have been already checked during the first iteration. By repeating this procedure, we obtain a minimizer of  $f$ . Since the first iteration requires  $O(n^2 + n^2 T_f) = O(n^2 T_f)$  time and all the other iterations require  $O(n + n T_f) = O(n T_f)$ , we have the following result.

**Theorem 4.3** *For  $d \leq 3$ , the posimodular function minimization can be solved in  $O(n^2 T_f)$  time.*

You might think that a similar property to Lemma 4.2 holds for a general  $d$ . However, the following instance indicates that this is not the case, since no nontrivial semi-extreme set is small. In fact, the size of each nontrivial semi-extreme set is independent of  $d$ .

**Example 4.4** Let  $S$  be an arbitrary subset of  $V$  with  $|S| \geq 4$ , and let  $f : 2^V \rightarrow \{0, 1, \dots, 7\}$  be a posimodular function defined as

$$f(X) = \begin{cases} 0 & \text{if } X = \emptyset, S \\ 1 & \text{if } X \subseteq S, |X| = 1 \text{ or } |S| - 1 \\ 2 & \text{if } X \subseteq S, 2 \leq |X| \leq |S| - 2 \\ 2 & \text{if } X \cap S = \emptyset, |X| = 1 \\ 3 & \text{if } X \cap S = \emptyset, |X| \geq 2 \\ 4 & \text{if } X \setminus S \neq \emptyset, |X \cap S| = 1 \\ 5 & \text{if } X \setminus S \neq \emptyset, 2 \leq |X \cap S| \leq |S| - 2 \\ 6 & \text{if } X \setminus S \neq \emptyset, |X \cap S| = |S| - 1 \\ 7 & \text{if } X \setminus S \neq \emptyset, X \cap S = S. \end{cases}$$

We note that  $\{S\} \cup \{\{v\} \mid v \in X\} \cup \{S \setminus \{v\} \mid v \in S\}$  is the family of all semi-extreme sets of  $f$ . Therefore, the size of each nontrivial semi-extreme set is either  $|S|$  or  $|S| - 1$ , which is independent of  $d$ . The posimodularity of  $f$  can be shown as follows.

Let  $X$  and  $Y$  be two subsets of  $V$  with  $X \cap Y \neq \emptyset$ . If both  $f(X) - f(X \setminus Y)$  and  $f(Y) - f(Y \setminus X)$  are nonnegative, then (1.1) clearly holds. We thus assume that at least



one pair  $(Z_1, Z_2)$  of  $(X \setminus Y, X)$  and  $(Y \setminus X, Y)$  satisfies one of the following conditions, where  $Z_1 \subseteq Z_2$ .

- (a)  $Z_1 \subseteq S$ ,  $|Z_1| = 1$ , and  $Z_2 = S$ .
- (b)  $Z_1 \subseteq S$ ,  $2 \leq |Z_1| \leq |S| - 2$ , and  $Z_2 = S$ .
- (c)  $Z_1 \subseteq S$ ,  $|Z_1| = |S| - 1$ , and  $Z_2 = S$ .
- (d)  $Z_1, Z_2 \subseteq S$ ,  $2 \leq |Z_1| \leq |S| - 2$ , and  $|Z_2| = |S| - 1$ .

If  $(Z_1, Z_2) = (X \setminus Y, X)$  satisfies (a) (i.e.,  $X = S$  and  $|X \setminus Y| = 1$ ), then we have  $f(X) = 0$ ,  $f(X \setminus Y) = 1$ ,  $|X \cap Y| = |S| - 1$ , and  $(Y \setminus X) \cap S = \emptyset$ . If  $Y \setminus X = \emptyset$ , then we have  $f(Y) = 1$  and  $f(Y \setminus X) = 0$ , which implies (1.1). On the other hand, if  $Y \setminus X \neq \emptyset$ , then we have  $f(Y) = 6$  and  $f(Y \setminus X) \leq 3$ , which again implies (1.1).

If  $(Z_1, Z_2) = (X \setminus Y, X)$  satisfies (b), then we have  $f(X) = 0$ ,  $f(X \setminus Y) = 2$ ,  $2 \leq |X \cap Y| \leq |S| - 2$ , and  $(Y \setminus X) \cap S = \emptyset$ . Hence if  $Y \setminus X = \emptyset$ , then it holds that  $f(Y) = 2$  and  $f(Y \setminus X) = 0$ . On the other hand, if  $Y \setminus X \neq \emptyset$ , then we have  $f(Y) = 5$  and  $f(Y \setminus X) \leq 3$ . In either case, (1.1) is derived.

If  $(Z_1, Z_2) = (X \setminus Y, X)$  satisfies (c), then we have  $f(X) = 0$ ,  $f(X \setminus Y) = 1$ ,  $|X \cap Y| = 1$ , and  $(Y \setminus X) \cap S = \emptyset$ . If  $Y \setminus X = \emptyset$ , then it holds that  $f(Y) = 1$  and  $f(Y \setminus X) = 0$ . On the other hand, if  $Y \setminus X \neq \emptyset$ , then  $f(Y) = 4$  and  $f(Y \setminus X) \leq 3$ . In either case, (1.1) is derived.

If  $(Z_1, Z_2) = (X \setminus Y, X)$  satisfies (d), then we have  $f(X) = 1$ ,  $f(X \setminus Y) = 2$ ,  $1 \leq |X \cap Y| \leq |S| - 3$ , and  $|(Y \setminus X) \cap S| \leq 1$ . Hence if  $Y \setminus X = \emptyset$ , then  $f(Y) \geq 1$  and  $f(Y \setminus X) = 0$ . If  $Y \setminus X \neq \emptyset$  and  $Y \setminus S = \emptyset$ , then  $f(Y) = 2$  and  $f(Y \setminus X) = 1$  by  $Y \subseteq S$ ,  $2 \leq |Y| \leq |S| - 2$ , and  $|Y \setminus X| = |(Y \setminus X) \cap S| = 1$ . If  $Y \setminus X \neq \emptyset$ ,  $Y \setminus S \neq \emptyset$ , and  $(Y \setminus X) \cap S = \emptyset$ , then  $f(Y) \geq 4$  and  $f(Y \setminus X) \leq 3$ . If  $Y \setminus X \neq \emptyset$ ,  $Y \setminus S \neq \emptyset$ , and  $(Y \setminus X) \cap S \neq \emptyset$ , then  $f(Y) \geq 5$  and  $f(Y \setminus X) = 4$  by  $|Y \cap S| = |Y \cap X| + |(Y \setminus X) \cap S| \geq 2$  and  $|(Y \setminus X) \cap S| = 1$ . In either case, (1.1) is derived.

## 4.2 Case in which $d$ is general

In this section, we propose an algorithm for the posimodular function minimization for general  $d$ . Different from our algorithm for  $d \leq 3$ , it is not based on the contraction for semi-extreme sets. Instead, we focus on the following simple property derived from posimodularity, and solve the problem by making use of dual Horn Boolean satisfiability problem.

**Lemma 4.5** *For a nonnegative integer  $d$ , let  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  be a posimodular function. If there exist a subset  $X$  of  $V$  and an element  $s \in V \setminus X$  such that*

$$f(X) \geq f(X \cup \{s\}), \quad (4.2)$$

*then any subset  $Y$  with  $Y \cap X = \emptyset$  satisfies  $f(Y) \geq f(Y \setminus \{s\})$ .*

**Proof.** If  $s \notin Y$ , then we clearly have  $f(Y) = f(Y \setminus \{s\})$ . On the other hand, if  $v \in Y$ , then by (1.1), we have  $f(X \cup \{s\}) + f(Y) \geq f(X) + f(Y \setminus \{s\})$ , which proves the lemma.  $\square$

Let us consider computing a locally minimal minimizer  $X^*$  of  $f$ . Here a subset  $X^*$  is called *locally minimal* if  $f(X^*) < f(X^* \setminus \{v\})$  holds for any  $v \in X^*$ . We note that a locally minimal minimizer  $X^*$  always exists if no singleton  $\{v\}$  is a minimizer of  $f$ , and such an  $X^*$  satisfies  $|X^*| \geq 2$ .

Let  $X$  be a subset of  $V$ , and let  $s$  be an element in  $V \setminus X$  that satisfies (4.2). Then Lemma 4.5 implies that any locally minimal subset  $X^*$  must satisfy  $s \notin X^*$  whenever  $X^* \cap X = \emptyset$ . To represent it as a Boolean formula, let us introduce propositional variables  $x_v$ ,

$v \in V$ , and we regard a Boolean vector  $x \in \{0, 1\}^V$  as a subset  $S_x$  such that  $S_x = \{v \in V \mid x_v = 1\}$ , i.e.,  $x$  is the characteristic vector of  $S_x$ . Then it can be represented as

$$x_s = 0 \text{ whenever } x_v = 0 \text{ for all } v \in X, \quad (4.3)$$

which is equivalent to satisfying the following dual Horn clause

$$\bigvee_{v \in X} x_v \vee \bar{x}_s. \quad (4.4)$$

If you have many pairs of  $X$  and  $s$  that satisfy (4.2), then their corresponding rules (4.4) reduce the search space for finding a locally minimal minimizer of  $f$ . Note that the rules can be represented as a dual Horn CNF, and hence the satisfiability can be solved in linear time and all satisfiable assignments can be generated with linear delay (i.e., the time interval between two consecutive output is bounded in linear time (in the input size)) [16]. However, the number of such pairs are in general exponential in  $n$ , and hence we need to find a subfamily  $\mathcal{P}$  of such pairs  $(X, s)$  such that (1) the size  $|\mathcal{P}|$  is polynomial in  $n$  (for a constant  $d$ ) and (2) the corresponding dual-Horn CNF has polynomially many satisfiable assignments.

**Definition 4.6** Let  $X$  be a subset of  $V$  with  $k = |X|$ . We say that  $X$  is *reachable* (from  $\emptyset$ ) if there exists a chain  $X_0 (= \emptyset) \subsetneq X_1 \subsetneq \dots \subsetneq X_k (= X)$  such that  $f(X_i) > f(X_{i-1})$  for all  $i = 1, 2, \dots, k$ , and *unreachable* otherwise.

By definition,  $\emptyset$  is reachable. An unreachable set  $U$  is called *minimal* if any proper subset of it is reachable. Let  $\mathcal{U}$  be the family of minimal unreachable sets  $U$ . From the definition of reachability, we have the following lemma.

**Lemma 4.7** For any minimal unreachable set  $U \in \mathcal{U}$ , we have  $f(U) \leq f(U \setminus \{u\})$  for all  $u \in U$ .

**Proof.** By definition,  $U \setminus \{u\}$  is reachable for all  $u \in U$ . Hence, if  $f(U) > f(U \setminus \{u\})$  holds for some  $u \in U$ , then it turns out that  $U$  is reachable, which is a contradiction.  $\square$

**Lemma 4.8** Let  $X^*$  be a locally minimal subset of a posimodular function  $f$ . Then the characteristic vector of  $X^*$  satisfies the dual Horn CNF  $\varphi_f$  defined by

$$\varphi_f = \bigwedge_{U \in \mathcal{U}} \bigwedge_{s \in U} \left( \bigvee_{u \in U \setminus \{s\}} x_u \vee \bar{x}_s \right) \quad (4.5)$$

**Proof.** Lemma 4.7, together with the discussion after Lemma 4.5 implies the lemma.  $\square$

Based on the lemma, we have the following algorithm for the posimodular function minimization.

**Algorithm** MINPOSIMODULAR( $f$ )

**Step 1.** Compute a singleton  $\{v^*\}$  with minimum  $f(v^*)$  (i.e.,  $f(v^*) = \min\{f(v) \mid v \in V\}$ ).

**Step 2.** Compute a subset  $S_{x^*}$  with minimum  $f(S_{x^*})$  among the sets  $S_x$  such that  $|S_x| \geq 2$  and  $\varphi_f(x) = 1$ .

**Step 3.** Output  $\{v^*\}$ , if  $f(v^*) \leq f(S_{x^*})$ , and  $S_{x^*}$ , otherwise. Halt.  $\square$

In the remaining part of this section, we show that  $\varphi_f$  has polynomially many clauses and satisfiable assignments in  $n$  (if  $d$  is bounded by a constant).

We first show basic facts for minimal unreachable sets, where a subset  $I$  of  $V$  is called *independent* of  $\mathcal{U}$  if it contains no  $U \in \mathcal{U}$ .

**Lemma 4.9** For a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ , we have the following three statements.

- (i)  $1 \leq |U| \leq d + 1$  holds for all  $U \in \mathcal{U}$ .
- (ii)  $|I| \leq d$  holds for all independent sets  $I$  of  $\mathcal{U}$ .
- (iii) If a singleton  $\{u\}$  is contained in  $\mathcal{U}$ , then  $f(u) = 0$ , and hence  $\{u\}$  is a minimizer of  $f$ .

**Proof.** Since  $\{0, 1, \dots, d\}$  is the range of  $f$ , any reachable set  $R$  has cardinality  $|R|$  at most  $d$ . This implies that (i) and (ii). (iii) follows from  $f(\emptyset) = 0$  by our assumption.  $\square$

Lemma 4.9 (i) implies that  $|\mathcal{U}| = O(\binom{n}{d+1})$  if  $d < n/2$ , and  $O(\binom{n}{n/2})$  otherwise. Hence we have

$$|\mathcal{U}| = O(n^{d+1}/d). \quad (4.6)$$

Let us then analyze the number of satisfiable assignments of  $\varphi_f$ . In order to make the discussion simpler, consider a definite Horn CNF  $\varphi_f(\bar{x})$ , where  $\bar{x}$  denotes the complement of  $x$ . Notice that a subset  $S_x$  with  $\varphi_f(\bar{x}) = 1$  is a candidate of the complement of a locally minimal minimizer of  $f$ . For a definite Horn CNF  $\varphi$  and a subset  $T$  of  $V$ , the following algorithm called *forward chaining procedure* (FCP) has been proposed to compute satisfiable assignments of  $\varphi$  [2, 6].

**Procedure** FCP( $\varphi; T$ )

**Step 0.** Let  $Q := T$ .

**Step 1.** While there exists a clause  $c$  in  $\varphi$  such that  $N(c) \subseteq Q$  and  $P(c) \cap Q = \emptyset$  do  
 $Q := Q \cup P(c)$ .

**Step 2.** Output  $Q$  as FCP( $\varphi; T$ ), and halt.

It is not difficult to see that  $T \subseteq FCP(\varphi; T)$  holds for any subset  $T$ , and  $FCP(\varphi; T) \subseteq FCP(\varphi; T')$  holds if  $T \subseteq T'$ . Moreover, for a definite Horn CNF  $\varphi$ , it is known [2, 6] that  $T$  corresponds to a satisfiable assignment of  $\varphi$  (i.e., the characteristic vector of  $T$  is a satisfiable assignment of  $\varphi$ ) if and only if  $T = FCP(\varphi; T)$ . This implies that for any subset  $T$ ,  $FCP(\varphi; T)$  corresponds to a satisfiable assignment of  $\varphi$ , and for any satisfiable assignment  $\alpha$  of  $\varphi$ , there exists a subset  $T$  such that  $FCP(\varphi; T)$  corresponds to  $\alpha$  (i.e.,  $S_\alpha = FCP(\varphi; T)$ ).

We now claim that for any satisfiable assignment  $\alpha$  of  $\varphi_f(\bar{x})$ , there exists a subset  $T$  such that  $|T| \leq d$  and  $S_\alpha = FCP(\varphi_f(\bar{x}); T)$ , which implies the number of satisfiable assignments of  $\varphi_f$  is bounded by  $\sum_{i=0}^d \binom{n}{i}$ .

For a satisfiable assignment  $\alpha$  of  $\varphi(\bar{x})$ , let  $\mathcal{U}_\alpha = \{U \subseteq \mathcal{U} \mid U \subseteq S_\alpha\}$ , and let  $I \subseteq S_\alpha$  be an independent set of  $\mathcal{U}_\alpha$  which is maximal in  $S_\alpha$  (i.e.,  $I \cup \{v\}$  is dependent of  $\mathcal{U}_\alpha$  for all  $v \in S_\alpha \setminus I$ ). Since  $\emptyset$  is independent of  $\mathcal{U}_\alpha$ , such an  $I$  must exist.

**Lemma 4.10** For a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ , let  $\alpha$  be a satisfiable assignment of  $\varphi_f(\bar{x})$ . Let  $I$  be defined as above. Then we have  $S_\alpha = FCP(\varphi_f(\bar{x}); I)$ .

**Proof.** If  $I = S_\alpha$ , we have  $S_\alpha = FCP(\varphi_f(\bar{x}); S_\alpha)$ , since  $\alpha$  is a satisfiable assignment of  $\varphi_f(\bar{x})$ . Assume that  $S_\alpha \setminus I$  is not empty. Then for each element  $v \in S_\alpha \setminus I$ ,  $I \cup \{v\}$  is dependent of  $\mathcal{U}_\alpha$ , i.e., some  $U \in \mathcal{U}_\alpha$  satisfies  $U \setminus I = \{v\}$ . This implies that  $\varphi_f(\bar{x})$  contains a clause  $c$  such that  $P(c) = \{v\}$  and  $N(c) = U \setminus \{v\} (\subseteq I)$ . Thus FCP( $\varphi_f(\bar{x}); I$ ) contains  $v$  for all  $v \in S_\alpha \setminus I$ , which implies  $S_\alpha \subseteq FCP(\varphi_f(\bar{x}); I)$ . Since  $I \subseteq S_\alpha$  and  $\alpha$  is satisfiable for  $\varphi_f(\bar{x})$ , we have  $S_\alpha = FCP(\varphi_f(\bar{x}); I)$ .  $\square$

**Lemma 4.11** For a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ , it holds that  $|\{x \in \{0, 1\}^n \mid \varphi_f(x) = 1\}| \leq \sum_{i=0}^d \binom{n}{i} (= O(n^d))$ .

**Proof.** By Lemma 4.10, for each satisfiable assignment  $\bar{\alpha}$  of  $\varphi_f$ , we have an independent set  $I$  of  $\mathcal{U}_\alpha$  such that  $S_\alpha = FCP(\varphi_f(\bar{\alpha}); I)$ . Since  $I$  is also independent of  $\mathcal{U}$ ,  $|I| \leq d$  holds by Lemma 4.9, which completes the proof.  $\square$

**Remark 4.12** Lemma 4.10 indicates that Step 2 of  $\text{MINPOSIMODULAR}(f)$  can be executed by applying FCP for all subsets  $T$  of  $V$  with  $|T| \leq d$ . For each  $T$ ,  $FCP(\varphi_f(\bar{x}); T)$  can be computed from  $\mathcal{U}$  in  $O(d|\mathcal{U}|) = O(n^{d+1})$  time. Thus, after computing  $\mathcal{U}$ , Step 2 of  $\text{MINPOSIMODULAR}(f)$  can be implemented to run in  $O(n^d(n^{d+1} + T_f)) = O(n^{2d+1} + n^d T_f)$  time.

Summarizing the arguments given so far, we have the following theorem.

**Theorem 4.13** For general  $d$ , the posimodular function minimization can be solved in  $O(n^d T_f + n^{2d+1})$  time.

**Proof.** Let us analyze the complexity of  $\text{MINPOSIMODULAR}(f)$ . Clearly, Steps 1 and 3 can be executed in  $O(n T_f)$  and  $O(n)$  time, respectively. As for Step 2,  $\mathcal{U}$  can be computed in  $O(n^d T_f + n^{d+1})$  time. Here we remark that it is not necessary to query the value of  $f(U)$  for any  $U \subseteq V$  with  $|U| = d + 1$ , if we know  $f(W)$  for all  $W \subseteq V$  with  $|W| \leq d$ . This together with Remark 4.12 implies that Step 2 requires  $O(n^d T_f + n^{2d+1})$  time.

Therefore, in total,  $\text{MINPOSIMODULAR}(f)$  requires  $O(n^d T_f + n^{2d+1})$  time.  $\square$

### 4.3 Corollaries of our algorithmical results

Let us first consider generating all minimizers of a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ . Note that  $f$  might have exponentially many minimizers. In fact, if  $f = 0$ , then we have  $2^n - 1$  minimizers. We thus consider output sensitive algorithms for it.

It follows from Lemma 4.8 that  $\text{MINPOSIMODULAR}(f)$  finds all locally minimal minimizers of  $f$ . Let  $S$  be a minimizer of  $f$  which is not locally minimal. By definition of locally minimality, there exists a chain  $T_0 (= T) \subsetneq T_1 \subsetneq \dots \subsetneq T_k (= S)$  from some locally minimal minimizer  $T$  of  $f$  such that for all  $i = 1, \dots, k$ ,  $T_i$  is a minimizer of  $f$  and  $|T_i \setminus T_{i-1}| = 1$ . Therefore, after generating all locally minimal minimizers of  $f$ , we check whether  $T \cup \{v\}$  is a minimizer of  $f$  for each minimizer  $T$  of  $f$  and  $v \notin T$ . This implies that all (not only locally minimal) minimizers of  $f$  can be generated in  $O(n T_f)$  delay after applying  $\text{MINPOSIMODULAR}(f)$  once.

**Corollary 4.14** For a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ , we can generate all minimizers of  $f$  in  $O(n T_f)$  delay, after  $O(n^d T_f + n^{2d+1})$  time to compute the first minimizer of  $f$ .

We next show that the family  $\mathcal{X}(f)$  of all extreme sets can be obtained as an application of  $\text{MINPOSIMODULAR}$ .

Recall that a subset  $X$  of  $V$  is called *extreme* if every nonempty proper subset  $Y$  of  $X$  satisfies  $f(Y) > f(X)$ . By definition,  $\mathcal{X}(f)$  contains all singletons  $\{v\}$ ,  $v \in V$ , and any extreme set  $X$  with  $|X| \geq 2$  is locally minimal. This together with Lemma 4.8 implies that Algorithm  $\text{MINPOSIMODULAR}$  checks all possible candidates for extreme sets. By the following simple observation, we only check the extremeness among such candidates.

**Lemma 4.15** *If a family  $\mathcal{Q} \subseteq 2^V$  contains all extreme sets of  $f$ , then  $X \in \mathcal{Q}$  is extreme for  $f$  if and only if any nonempty proper subset  $Y$  of  $X$  with  $Y \in \mathcal{Q}$  satisfies  $f(Y) > f(X)$ .*

**Proof.** If some nonempty proper subset  $Y$  of  $X$  with  $Y \in \mathcal{Q}$  satisfies  $f(Y) \leq f(X)$ , then  $X$  is not extreme for  $f$ . On the other hand, if  $X$  is not extreme, then some nonempty proper subset  $Y$  of  $X$  satisfies  $f(Y) \leq f(X)$ . If  $Y$  is not contained in  $\mathcal{Q}$ , then  $Y$  is not extreme for  $f$ , and hence there exists an extreme set  $Z$  of  $f$  such that  $Z \subseteq Y$  and  $f(Z) \leq f(Y)$ . Note that this  $Z$  is a nonempty proper subset of  $X$  with  $f(Z) \leq f(X)$ , which is contained in  $\mathcal{Q}$ .  $\square$

**Algorithm** COMPUTEEXTREMESSETS( $f$ )

**Step 1.** Let  $\mathcal{X} := \emptyset$  and let  $\mathcal{Q} := \{v \mid v \in V\} \cup \{V \setminus FCP(\varphi_f(\bar{x}); I) \mid I \subseteq V, |I| \leq d\}$ .

/\* Here all  $f(X)$ ,  $X \in \mathcal{Q}$  are assumed to be stored. \*/

**Step 2.** For each  $X \in \mathcal{Q}$  do

If all nonempty  $Y \in \mathcal{Q}$  with  $Y \subsetneq X$  satisfy  $f(Y) > f(X)$ , then  $\mathcal{X} := \mathcal{X} \cup \{X\}$ .

Output  $\mathcal{X}$  (as  $\mathcal{X}(f)$ ) and halt.  $\square$

Similarly to Algorithm MINPOSIMODULAR, Step 1 requires  $O(n^d T_f + n^{2d+1})$  time. Moreover, by  $|\mathcal{Q}| = O(n^d)$ , Step 2 can be executed in  $O(n^{2d+1})$  time.

In summary, we have the following result.

**Corollary 4.16** *For a posimodular function  $f : 2^V \rightarrow \{0, 1, \dots, d\}$ , we can compute the family  $\mathcal{X}(f)$  of all extreme sets of  $f$  in  $O(n^d T_f + n^{2d+1})$  time.*

## 5 Posimodular function maximization

In this section, we consider the posimodular function maximization defined as follows.

POSIMODULAR FUNCTION MAXIMIZATION

Input: A posimodular function  $f : 2^V \rightarrow \mathbb{R}_+$ , (5.1)

Output: A nonempty subset  $X$  of  $V$  maximizing  $f$ .

Here we assume that the optimal value  $f(X^*)$  is also output. Similarly to the posimodular function minimization, the problem (5.1) is in general intractable.

**Theorem 5.1** *Any algorithm for the posimodular function maximization requires at least  $2^{n-1}$  oracle calls.*

**Proof.** Let us first consider the case in which  $n$  is even, i.e.,  $n = 2k$  for some positive integer  $k$ . Let  $g : 2^V \rightarrow \mathbb{R}_+$  be a function defined by  $g(X) = |X|$  if  $|X| \leq k - 1$ , and  $g(X) = k$  otherwise, and for a subset  $S \subseteq V$  with  $|S| \geq k$ , define a function  $g_S : 2^V \rightarrow \mathbb{R}_+$  by  $g_S(X) = g(X)$  if  $X \neq S$ , and  $g_S(X) = k + 1$  if  $X = S$ . Since  $g$  is monotone, it is posimodular. We claim that  $g_S$  is also posimodular.

Note that  $g_S(Z) \geq g_S(Z')$  holds for any pair of subsets  $Z$  and  $Z'$  with  $Z \supseteq Z'$  except for  $Z' = S$ . Let  $X$  and  $Y$  be two subsets of  $V$  with  $X \cap Y \neq \emptyset$ . In order to check the posimodular inequality (1.1), we can assume that  $S = X \setminus Y$  or  $Y \setminus X$ , since all the other cases can be proven easily. By symmetry, let  $S = X \setminus Y$ . Then we have  $g_S(X) = k$ ,  $g_S(X \setminus Y) = k + 1$ , and since  $|Y \setminus X| \leq n - k - 1 = k - 1$ ,  $g_S(Y) > g_S(Y \setminus X)$  holds. These imply the posimodular inequality.

Let  $q = \sum_{i=k}^n \binom{n}{i} (\geq 2^{n-1})$ . Assume that there exists an algorithm  $A$  for the posimodular function maximization which requires oracle calls smaller than  $q$ . Let  $\mathcal{X}$  denote the family

of subsets of  $V$  which are called by  $A$  if a posimodular function  $g$  is an input of  $A$ . Since  $|\mathcal{X}| \leq q-1$ , we have a subset  $S$  such that  $S \notin \mathcal{X}$  and  $|S| \geq k$ . This implies that  $g_S(X) = g(X)$  for all  $X \in \mathcal{X}$ , which contradicts that Algorithm  $A$  distinguishes between  $g$  and  $g_S$  (i.e.,  $A$  cannot know if the optimal value is either  $k$  or  $k+1$ ).

Next let us consider the case in which  $n$  is odd, i.e.,  $n = 2k+1$  for some nonnegative integer  $k$ . Let  $g : 2^V \rightarrow \mathbb{R}_+$  be a function defined by  $g(X) = |X|$  if  $|X| \leq k$ , and  $g(X) = k+1$  otherwise, and for a subset  $S \subseteq V$  with  $|S| \geq k+1$ , define a function  $g_S : 2^V \rightarrow \mathbb{R}_+$  by  $g_S(X) = g(X)$  if  $X \neq S$ , and  $g_S(X) = k+2$  if  $X = S$ . In a similar way to the previous case, we can observe that at least  $\sum_{i=k+1}^n \binom{n}{i} \geq 2^{n-1}$  oracle calls are required to solve the posimodular function maximization.  $\square$

Next consider the case where  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  for a nonnegative integer  $d$ . Then we have the following tight result for the posimodular function maximization.

**Theorem 5.2** *The posimodular function maximization for  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  with a constant  $d$  can be solved in  $\Theta(n^{d-1}T_f)$  time.*

The following lemma shows the lower bound for the posimodular function maximization, where the upper bound will be shown in the next subsection.

**Lemma 5.3** *The posimodular function maximization for  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  requires  $\Omega(n^{d-1})$  oracle calls, if  $n \geq 2d-2$ .*

**Proof.** Let  $g : 2^V \rightarrow \{0, 1, \dots, d\}$  be a function defined by  $g(X) = |X|$  if  $|X| \leq d-2$ , and  $g(X) = d-1$  otherwise. For a subset  $S \subseteq V$  with  $|S| \geq n-d+1$  ( $\geq d-1$ ), define a function  $g_S : 2^V \rightarrow \{0, 1, \dots, d\}$  by  $g_S(X) = g(X)$  if  $X \neq S$ , and  $g_S(X) = d$  if  $X = S$ . Since  $g$  is monotone, it is posimodular. We claim that  $g_S$  is also posimodular.

Note that  $g_S(Z) \geq g_S(Z')$  holds for any pair of subsets  $Z$  and  $Z'$  with  $Z \supseteq Z'$  except for  $Z' = S$ . Let  $X$  and  $Y$  be two subsets of  $V$  with  $X \cap Y \neq \emptyset$ . In order to check the posimodular inequality (1.1), we can assume that  $S = X \setminus Y$  or  $Y \setminus X$ , since all the other cases can be proven easily. By symmetry, let  $S = X \setminus Y$ . Then we have  $g_S(X) = d-1$ ,  $g_S(X \setminus Y) = d$ , and since  $|Y \setminus X| \leq n - |S| - 1 \leq d-2$  and  $|Y| > |Y \setminus X|$ ,  $g_S(Y) > g_S(Y \setminus X)$  holds. These imply the posimodular inequality.

In a similar way to the proof of Theorem 5.1, we can observe that the posimodular function maximization requires at least  $\sum_{i=n-d+1}^n \binom{n}{i} = \Omega(n^{d-1})$  oracle calls, to distinguish among  $g$  and all  $g_S$  with  $|S| \geq n-d+1$ .  $\square$

## 5.1 Polynomial time algorithm for a constant $d$

In this section, we present an  $O(n^{d-1}T_f)$ -time algorithm for the posimodular function maximization for a constant  $d$ .

The following simple lemma implies that the problem can be solved in  $O(n^dT_f)$  time.

**Lemma 5.4** *Let  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  be a posimodular function, and let  $S$  be a maximal maximizer of  $f$  (i.e., a maximizer such that no proper superset is a maximizer of  $f$ ). Then,  $f(X \cup \{v\}) \geq f(X) + 1$  holds for any pair of a set  $X \subseteq V$  and an element  $v \in V$  such that  $X$ ,  $\{v\}$  and  $S$  are pairwise disjoint.*

**Proof.** By (1.1), we have  $f(X \cup \{v\}) + f(S \cup \{v\}) \geq f(X) + f(S)$ . By the maximality of  $S$ , we have  $f(S \cup \{v\}) < f(S)$ . Hence, we have  $f(X \cup \{v\}) > f(X)$ .  $\square$

**Corollary 5.5** *Let  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  be a posimodular function. Then we have  $|S| \geq n - d$  for any maximal maximizer  $S$  of  $f$ .*

**Proof.** Let  $k = |S|$ , and let  $X_0 (= \emptyset) \subseteq X_1 \subseteq \dots \subseteq X_{n-k} (= V \setminus S)$  be a chain with  $|X_i| = i$  for all  $i$ . Then it follows from Lemma 5.4 that

$$f(X_0) (= 0) < f(X_1) < \dots < f(X_{n-k}) (\leq d), \quad (5.2)$$

which implies that  $n - k \leq d$ .  $\square$

By the corollary, the posimodular function maximization can be solved in  $O(n^d T_f)$  time by checking all subsets  $X$  with  $|X| \geq n - d$ .

In the remaining part of this section, we reduce the complexity to  $O(n^{d-1} T_f)$  by showing a series of lemmas which assumes that no maximizer of  $f$  has size at least  $n - d + 1$ , i.e., by Corollary 5.5 and (5.2),

$$\text{any maximal maximizer } X^* \text{ of } f \text{ satisfies } |X^*| = n - d \text{ and } f(X^*) = d. \quad (5.3)$$

By (5.2), it implies that  $n \geq 2d$ .

**Lemma 5.6** *Under the assumption (5.3), we have the following two statements.*

- (i) *For any maximizer  $S$  of  $f$ , there exists a maximizer  $S'$  of  $f$  with  $S' \cap S = \emptyset$  and  $|S'| = d$ .*
- (ii) *Let  $S_1, S_2$  be two maximizers of  $f$  with  $S_1 \cap S_2 = \emptyset$ . Then, there exist two maximizers  $X_1, X_2$  of  $f$  with  $|X_1| = |X_2| = d$  and  $X_i \subseteq S_i$ ,  $i = 1, 2$ . Moreover, any subset  $Y \subseteq V$  with  $X_1 \subseteq Y \subseteq V \setminus X_2$  or  $X_2 \subseteq Y \subseteq V \setminus X_1$  is a maximizer of  $f$ .*

**Proof.** (i). Let  $S$  be an arbitrary maximizer of  $f$ , and  $S_1$  be a maximal maximizer of  $f$  with  $S_1 \supseteq S$ . By (5.3), we have  $|S_1| = n - d$  and hence  $|V \setminus S_1| = d$ . It follows from (5.2) that  $f(V \setminus S_1) = d$ , which means that  $V \setminus S_1$  is a maximizer of  $f$  with size  $d$  which is disjoint from  $S$ .

(ii). Since we have  $f(V \setminus S_1) + f(V \setminus S_2) \geq f(S_1) + f(S_2)$  by (1.1), both  $V \setminus S_1$  and  $V \setminus S_2$  are also maximizers of  $f$ . By (5.3), we have  $|V \setminus S_j| \leq n - d$  and  $|S_j| \geq d$  for  $j = 1, 2$ . By applying (i) to  $V \setminus S_j$  ( $j = 1, 2$ ), we obtain a maximizer  $X_j \subseteq S_j$  with  $|X_j| = d$ . Here we note that  $X_1 \cap X_2 = \emptyset$ . Moreover, for any set  $Z \subseteq V \setminus (X_1 \cup X_2)$ , both  $X_1 \cup Z$  and  $X_2 \cup Z$  are also maximizers of  $f$ , since we have  $f(X_1 \cup Z) + f(X_2 \cup Z) \geq f(X_1) + f(X_2)$  by (1.1). This completes the proof.  $\square$

**Lemma 5.7** *Assume that (5.3) holds. Let  $S$  be a maximizer of  $f$  with size  $d$ , and let  $X$  be a subset of  $V$  such that  $|X| = f(X) = d - 1$  and  $X \cap S = \emptyset$ . Then, there exists an element  $v \in V \setminus (S \cup X)$  with  $f(X \cup \{v\}) = d$ .*

**Proof.** Let  $S'$  be a maximizer of  $f$  with  $S' \cap S = \emptyset$  and  $|S'| = d$  such that  $|S' \setminus X|$  is the minimum. We note that such an  $S'$  always exists by Lemma 5.6 (i), and  $S' \setminus X \neq \emptyset$  is satisfied by  $|S'| > |X|$ . Moreover, it follows from Lemma 5.6 (ii) that  $V \setminus (X \cup S')$  is also a maximizer of  $f$ . For  $v \in S' \setminus X$ , we have  $f(X \cup \{v\}) + f(V \setminus (X \cup (S' \setminus \{v\}))) \geq f(X) + f(V \setminus (X \cup S')) = 2d - 1$  by (1.1). Therefore, it suffices to show that  $f(V \setminus (X \cup (S' \setminus \{v\}))) \leq d - 1$  to prove  $f(X \cup \{v\}) = d$ .

Assume to the contrary that  $f(V \setminus (X \cup (S' \setminus \{v\}))) = d$ . By Lemma 5.6 (i), there exists a maximizer  $S''$  of  $f$  with  $|S''| = d$  and  $S'' \cap (V \setminus (X \cup (S' \setminus \{v\}))) = \emptyset$ , i.e.,  $S'' \subseteq X \cup (S' \setminus \{v\})$ , which contradicts the minimality of  $|S' \setminus X|$ .  $\square$

We remark that  $S$  and  $X$  in Lemma 5.7 always exist if (5.3) is satisfied. In fact, by Lemma 5.6, we have two maximizers  $X_1$  and  $X_2$  of  $f$  such that  $|X_1| = |X_2| = d$ ,  $X_1 \cap X_2 = \emptyset$ , and  $V \setminus X_2$  is also a maximizer of  $f$ . Let  $S = X_1$  and  $X = X_2 \setminus \{v\}$  for any  $v \in X_2$ . Then  $S$  satisfies the condition in Lemma 5.7, and since  $V \setminus X_2$  is a maximal maximizer of  $f$ , (5.2) implies that  $X$  also satisfies the condition in Lemma 5.7.

**Lemma 5.8** *Let  $\mathcal{X}$  be the family of all subsets  $X$  of  $V$  such that  $|X| = d - 1$  and  $X \cap S \neq \emptyset$  for all maximizers  $S$  of  $f$  with  $|S| = d$ . Then, under the assumption (5.3), we have  $|\mathcal{X}| = O(n^{d-3})$ .*

**Proof.** By Lemma 5.6, there exist two maximizers  $S_1$  and  $S_2$  of  $f$  with  $|S_1| = |S_2| = d$  and  $S_1 \cap S_2 = \emptyset$ . Clearly,  $|\mathcal{X}|$  is bounded by the number of sets  $X$  with size  $d - 1$  with  $X \cap S_1, X \cap S_2 \neq \emptyset$ , which is

$$\sum_{i,j>0, i+j \leq d-1} \binom{d}{i} \binom{d}{j} \binom{n-2d}{d-1-i-j} \leq \sum_{k=2}^{d-1} \binom{2d}{k} \binom{n-2d}{d-1-k} = O(n^{d-3}).$$

□

Let  $c$  be a constant such that  $|\mathcal{X}| \leq cn^{d-3}$  for  $\mathcal{X}$  in Lemma 5.8. Based on these lemmas, we can find a maximizer of  $f$  in the following manner:

**Algorithm** MAXPOSIMODULAR( $f$ )

**Step 1.** Find a subset  $X_1$  of  $V$  such that  $|X_1| \geq n - d + 1$  and  $f(X_1) = \max\{f(X) \mid X \subseteq V, |X| \geq n - d + 1\}$ . If  $f(X_1) = d$ , then output  $X_1$  and halt.

**Step 2.** Find a subset  $X_2$  of  $V$  such that  $|X_2| = d - 1$  and  $f(X_2) = \max\{f(X) \mid X \subseteq V, |X| = d - 1\}$ . If  $f(X_2) = d$ , then output  $X_2$  and halt. If  $f(X_2) \leq d - 2$ , then output  $X_1$  and halt.

**Step 3.** Choose  $\min\{cn^{d-3} + 1, |\mathcal{X}_1|\}$  members  $X$  from  $\mathcal{X}_1 = \{X \subseteq V \mid |X| = d - 1, f(X) = d - 1\}$ . For each such  $X$ , if  $f(X \cup \{v\}) = d$  for some  $v \notin X$ , then output  $X \cup \{v\}$  and halt.

**Step 4.** Output  $X_1$  and halt.

**Lemma 5.9** *Algorithm MAXPOSIMODULAR( $f$ ) solves the posimodular function minimization for  $f : 2^V \rightarrow \{0, 1, \dots, d\}$  for a constant  $d$  in  $O(n^{d-1}T_f)$  time.*

**Proof.** Let us first prove the correctness of the algorithm. Let  $S$  be a maximal maximizer of  $f$ . Assume that  $f(S) = d$  holds. Then Corollary 5.5 implies that  $|S| \geq n - d$ . If  $|S| \geq n - d + 1$ , then  $S$  can be found in Step 1. On the other hand, if  $|S| = n - d$ , then we have (5.3). By the discussion after Lemma 5.7,  $f(X_2) \geq d - 1$  must hold. If  $f(X_2) = d$ , then  $X_2$  is clearly a maximizer of  $f$  which is output in Step 2. Otherwise (i.e.,  $f(X_2) = d - 1$ ), by Lemma 5.7 together with the discussion after Lemma 5.7, for each subset  $X$  with  $|X| = f(X) = d - 1$ , we only check if  $f(X \cup \{v\}) = d$  for some  $v \notin X$ . Moreover, it follows from Lemma 5.8 that we only check at most  $cn^{d-3} + 1$  many such  $X$ . Therefore, in this case, Step 3 correctly outputs a maximizer of  $f$ .

Assume next that  $f(S) \leq d - 1$ . Then Algorithm MAXPOSIMODULAR( $f$ ) output  $X_1$  in Step 2 or 4, which is correct, since there exists a maximal maximizer of size at least  $n - d + 1$  by Corollary 5.5.

As for the time complexity of Algorithm MAXPOSIMODULAR( $f$ ), we see that Steps 1 and 2 can be executed in  $O(n^{d-1}T_f)$  time. Since Steps 3 and 4 respectively require  $O(n^{d-2}T_f)$  and  $O(n)$  time, in total, algorithm requires  $O(n^{d-1}T_f)$  time. □



**Remark 5.10** For  $\mathcal{X}$  defined in Lemma 5.8, we have  $|\mathcal{X}| = O(n^{d-2})$  if  $d = O(\sqrt{n})$ . As observed in the proof of Lemma 5.9, the time complexity of Algorithm MAXPOSIMODULAR( $f$ ) is  $O((n^{d-1} + n|\mathcal{X}|)T_f)$ . Hence, it follows that the posimodular function maximization has time complexity  $\Theta(n^{d-1}T_f)$  even for  $d = O(\sqrt{n})$ .

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